Normal Maths Pure Notes

Algebraic Expressions

$a^m \times a^n = a^{m+n}$, $a^m \div a^n = a^{m-n}$, $(a^n)^n = a^{mn}$, $(ab)^n = a^n b^n$

$x^2 - y^2 = (x+y)(x-y)$, $a^{\frac{1}{m}} = \sqrt[m]{a}$, $a^{-m} = \frac{1}{a^m}$, $a^{\frac{n}{m}} = \sqrt[m]{a^n}$, $a^0 = 1$

$\sqrt{ab} = \sqrt{a} \times \sqrt{b}$, $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$,

Fractions in the form $\frac{1}{\sqrt{a}}$ multiply by $\frac{\sqrt{a}}{\sqrt{a}}$

Fractions in the form $\frac{1}{a+\sqrt{b}}$ multiply by $\frac{a-\sqrt{b}}{a-\sqrt{b}}$

Fractions in the form $\frac{1}{a-\sqrt{b}}$ multiply by $\frac{a+\sqrt{b}}{a+\sqrt{b}}$

Quadratics

Quadratic formula: $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 + \left(c - \frac{b^2}{4a}\right.$

if $f(x) = ax^2 + bx + c = 0$

$b^2 - 4ac =$ 'discriminant'

$b^2 - 4ac < 0$: no real roots

$b^2 - 4ac = 0$: one repeated real root

$b^2 - 4ac > 0$: two distict real roots

if $f(x) = a(x+p)^2 + q$

there is a turning point at $(-p, q)$

Equations and Inequalities

The solutions of a pair of simultaneous equations represent the points of intersection of their graphs. If $y > f(x)$ or $y < f(x)$ the line is dotted, if $y \geqslant f(x)$ or $y \leqslant f(x)$ then the graph is a solid line.

Graphs and Transformations

The graphs $y = \frac{k}{x}$ and $y = \frac{k}{x^2}$ have asymptotes at $x = 0$ and $y = 0$

$y = f(x+a)$ = translation $-a$ in x-axis

$y = af(x)$ = stretch factor a in y-axis

$y = -f(x)$ reflection i[n] x-axis

$y = f(x) + a$ = translation a in y-axis

$y = f(ax)$ = stretch factor $\frac{1}{a}$ in x-axis

$y = f(-x)$ reflection in y-axis

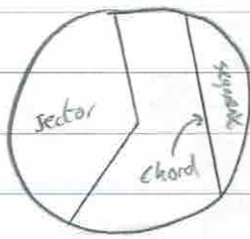
Straight-Line Graphs

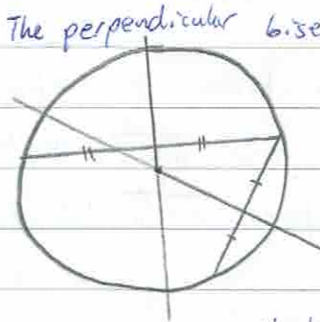
$y - y_1 = m(x - x_1)$ a perpendicular line $-\frac{1}{m}$

distance between two points: $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Circles

midpoint $= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$ circle: $(x-a)^2 + (y-b)^2 = r^2$





The perpendicular bisector of a chord will go through the centre of a circle.
To find the centre of a circle given 3 points find the two perpendicular bisectors of two chords and find where they meet.

Algebraic Methods

You can prove a statement is true by deduction. This is starting from known factors or definitions, then using logical steps to reach the desired conclusion.

In a proof:
- State any information or assumptions you are using
- Show every step of your proof clearly
- Make sure that every step follows from the last
- Make sure you have covered all possible cases
- Write a statement of proof at the end

To prove an identity:
- Start with the expression on one side of the identity
- Manipulate that expression algebraically until it matches the other side
- Show every step

You can prove a statement by exhaustion. This is breaking it into smaller cases and proving them separately.

You can prove a statement is not true by a counter-example.

The Binomial Expansion

Pascal's triangle

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

The $(n+1)$th row gives the coefficients in the expansion of $(a+b)^n$

^nC_r or $\binom{n}{r} = \frac{n!}{r!(n-r)!}$

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \dots + \binom{n}{r}a^{n-r}b^r + \dots + b^n$$

general term $= \binom{n}{r}a^{n-r}b^r$

Trigonometric Ratios

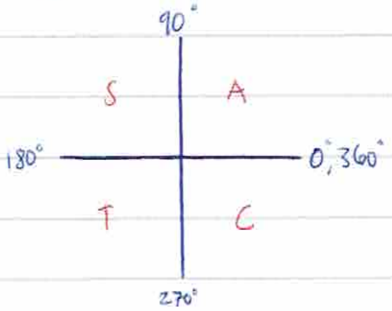
$a^2 = b^2 + c^2 - 2bc\cos A$

$\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$

$\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

Area $= \frac{1}{2}ab\sin C$

Trigonometric identities and equations



$\sin 30 = 1/2$ $\cos 30 = \frac{\sqrt{3}}{2}$ $\tan 30 = \frac{\sqrt{3}}{3}$

$\sin 45 = \sqrt{2}/2$ $\cos 45 = \frac{\sqrt{2}}{2}$ $\tan 45 = 1$

$\sin 60 = \sqrt{3}/2$ $\cos 60 = \frac{1}{2}$ $\tan 60 = \sqrt{3}$

$\sin^2\theta + \cos^2\theta \equiv 1$ $\tan\theta \equiv \dfrac{\sin\theta}{\cos\theta}$ principle value = angle the calc. gives

Vectors

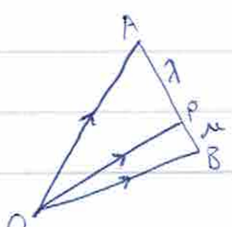
Any vector is equivalent to ~~adding a negative vector~~ $\lambda \underline{a}$ if it's parallel to $\underline{a}$

To multiply a column vector by a scalar multiply each component by the scalar. To add two column vectors add the x and y components

$\begin{pmatrix} p \\ q \end{pmatrix} + \begin{pmatrix} r \\ s \end{pmatrix} = \begin{pmatrix} p+r \\ q+s \end{pmatrix}$. $\begin{pmatrix} p \\ q \end{pmatrix} = pi + qj$

A unit vector is a vector of length 1. $|\underline{a}| = \sqrt{x^2+y^2}$

$\hat{\underline{a}} = \dfrac{\underline{a}}{|\underline{a}|}$



$\overrightarrow{OP} = \overrightarrow{OA} + \frac{\lambda}{\lambda+\mu}\overrightarrow{AB}$

$= \overrightarrow{OA} + \frac{\lambda}{\lambda+\mu}(\overrightarrow{OB} - \overrightarrow{OA})$

if $\underline{a}$ and $\underline{b}$ are non-parallel and

$p\underline{a} + q\underline{b} = r\underline{a} + s\underline{b}$

$p = r$, $q = s$

Differentiation

The gradient of a curve at a given point is defined as the gradient of the tangent to the curve at that point.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \qquad y = ax^n, \quad \frac{dy}{dx} = anx^{n-1}$$

The function $f(x)$ is increasing if $f'(x) \geq 0$
The function $f(x)$ is decreasing if $f'(x) \leq 0$

Type of stationary point	$f'(x-h)$	$f'(x)$	$f'(x+h)$
Local maximum	+	0	−
Local min	−	0	+
Point of inflection	+	0	+
	−	0	−

or $f''(a) = 0$: max/min/inflect.

if $f''(a) > 0$: ~~maximum~~ minimum $\quad f''(a) < 0$: maximum

Integration

Area $= \int_a^b y \cdot dx$ (will be negative if below x-axis)

Exponentials and Logarithms

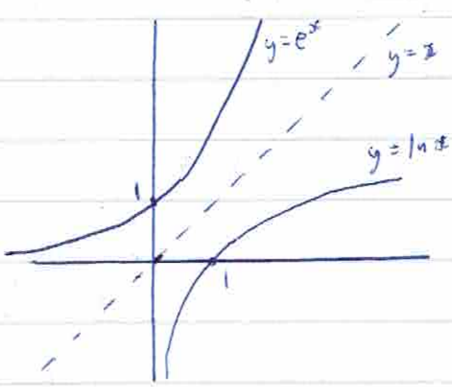
if $y = e^{kx}$
$\frac{dy}{dx} = ke^{kx}$

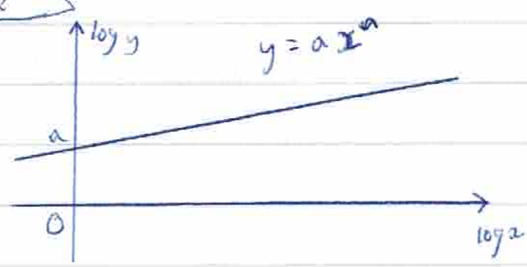
if $\log_a n = x$
$n = a^x$

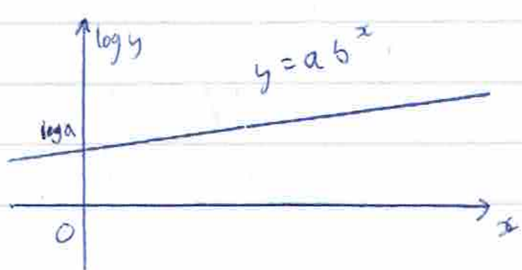
$\log x + \log y = \log xy$
$\log x - \log y = \log \frac{x}{y}$
$\log(x^k) = k \log x$

$\log_a a = 1$
$\log_a 1 = 0$

$$\boxed{e^{\ln x} = x}$$







A-level

Algebraic Methods

To prove a statement by contradiction, you start by assuming it is not true. You then use logical steps to show that this leads to something impossible.
↳ rational number = $\frac{a}{b}$ where a and b are integers with no common factor.

A fraction with two+ linear factors in the denominator can be split into two+ separate fractions. This is called splitting it into partial fractions. If the fraction is proper (numerator has a degree less than the denominator) it can be split into partial fractions of a constant over one of the linear factors. If it is improper (numerator has a degree equal to or larger than the denominator), you can either convert it to a mixed fraction using algebraic division or $F(x) = Q(x) \times \text{divisor} + \text{remainder}$ or find the form of the partial fractions. This can be done by looking at the degrees of the numerator and denominator. If they are equal, add A to the partial fractions, if the numerator is ~~one~~ one greater add $Ax + B$ ....etc.

Functions and Graphs

When $f(x) \geq 0$, $|f(x)| = f(x)$. When $f(x) < 0$, $|f(x)| = -f(x)$.
If drawing the graph of $|f(x)|$, reflect all values below the x-axis in the x-axis. If drawing $f(|x|)$, reflect all values where $x > 0$ in the y-axis (still keeping them on the right side).
A function must have one output for one input, can have many inputs for same output.
$fg(x)$ means apply g first, then f.
$f(x)$ is the inverse of $f^{-1}(x)$, $ff^{-1}(x) = f^{-1}f(x) = x$
$f^{-1}(x)$ is the reflection of $f(x)$ in the line $y = x$.
Domain of $f(x)$ = range of $f^{-1}(x)$, Domain of $f^{-1}(x)$ = range of $f(x)$

$f(x+a)$ = horizontal translation of $-a$
$f(x) + a$ = vertical translation of $+a$
$f(ax)$ = horizontal stretch of $\frac{1}{a}$
$af(x)$ = vertical stretch of a
$f(-x)$ = reflection in y-axis
$-f(x)$ = reflection in x-axis

Sequences and Series

Arithmetic: difference between terms is constant

$U_n = a + (n-1)d$

$S_n = \frac{n}{2}(2a + (n-1)d)$

$= \frac{n}{2}(a + l)$

proof: write out terms, flip, find $2S_n$, find S_n

Geometric: common ratio between consecutive terms

$U_n = ar^{n-1}$

$S_n = \frac{a(1-r^n)}{1-r} = \frac{a(r^n - 1)}{r-1}$

proof: write out terms, write out terms of rS_n, find $S_n - rS_n$, divide by $(1-r$

$S_\infty = \frac{a}{1-r} \quad (|r| < 1)$

A sequence is:

increasing if $u_{n+1} > u_n$

decreasing if $u_{n+1} < u_n$

periodic if it cycles: if $U_{n+k} = U_n$ (k = 'order' of the sequence)

Binomial Expansion

If n is negative or a fraction use: $(1+x)^n = 1 + nx + \frac{n(n-1)x^2}{2!} \ldots$

valid: $|x| < 1$

if given $(a+bx)^n$

do $= (a(1+\frac{b}{a}x))^n = a^n(1+\frac{b}{a}x)^n$

Radians

$\sin\frac{\pi}{6} = 1/2$	$\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$	$\tan\frac{\pi}{6} = \frac{\sqrt{3}}{3}$
$\sin\frac{\pi}{4} = \sqrt{2}/2$	$\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$	$\tan\frac{\pi}{4} = 1$
$\sin\frac{\pi}{3} = \sqrt{3}/2$	$\cos\frac{\pi}{3} = 1/2$	$\tan\frac{\pi}{6} = \sqrt{3}$

Arc length = $L = r\theta$

Area sector = $A = \frac{1}{2}r^2\theta$

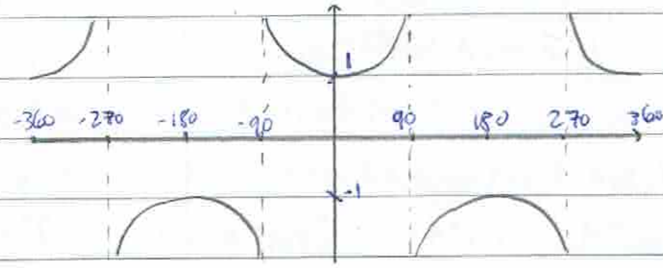
Area segment = $A = \frac{1}{2}r^2(\theta - \sin\theta)$

when θ is small and in radians: $\sin\theta \approx \theta$ $\quad \tan\theta \approx \theta$ $\quad \cos\theta \approx 1 - \frac{\theta^2}{2}$

Trigonometric Functions

$\sec x = \frac{1}{\cos x}$, $\operatorname{cosec} x = \frac{1}{\sin}$, $\cot x = \frac{1}{\tan x} = \frac{\cos x}{\sin x}$

$y = \sec x$

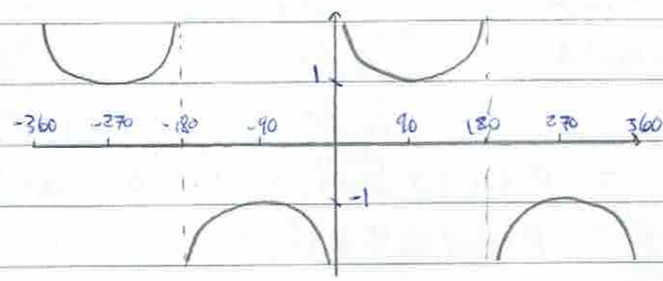


$\sin^2 x + \cos^2 x \equiv 1$

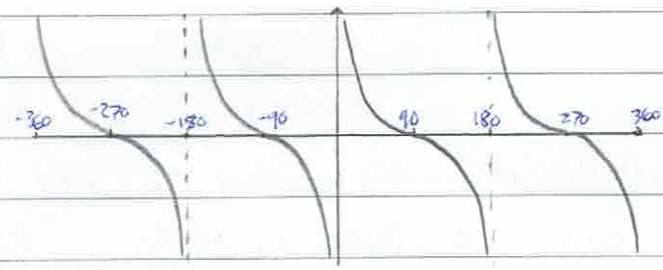
$1 + \tan^2 x \equiv \sec^2 x$

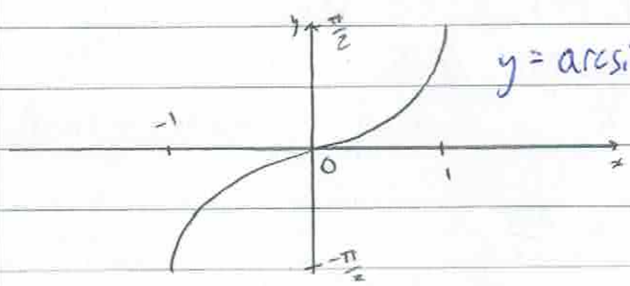
$1 + \cot^2 x \equiv \operatorname{cosec}^2 x$

$y = \operatorname{cosec} x$

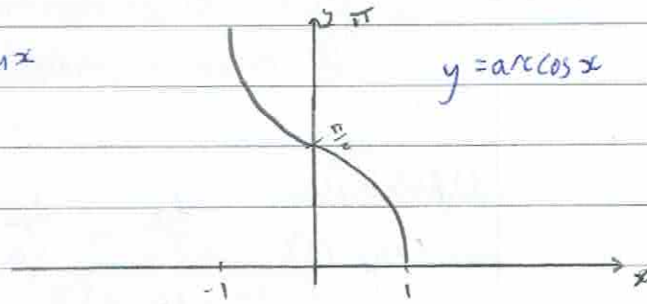


$y = \cot x$

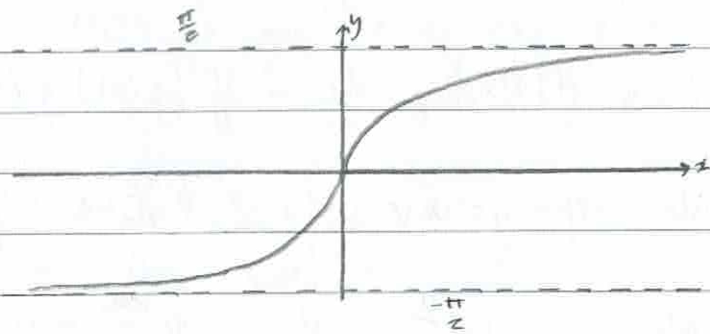




$y = \arcsin x$



$y = \arccos x$



$y = \arctan x$

Trigonometry and Modelling

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \cos B$$
$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$
$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\sin 2A = 2 \sin A \cos A$$
$$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$$
$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$$

$$a \sin x \pm b \cos x = R \sin(x \pm \alpha) \quad (R > 0, \ 0 < \alpha < 90^\circ)$$
$$a \cos x \pm b \sin x = R \cos(x \mp \alpha)$$
$$R = \sqrt{a^2 + b^2}$$

Parametric Equations

A curve can be defined using parametric equations $x = p(t)$ and $y = q(t)$.
You can convert to cartesian by using substitution.
if $x = p(t)$ and $y = q(t)$ and $y = f(x)$
$f(x)$ domain = range of $p(t)$
$f(x)$ range = range of $q(t)$

Differentiation

Chain rule: $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

if $y = (f(x))^n$, $\frac{dy}{dx} = n f'(x) (f(x))^{n-1}$

if $y = f(g(x))$, $\frac{dy}{dx} = f'(g(x)) g'(x)$

Product rule: if $y = uv$, $\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$

Quotient rule: if $y = \frac{u}{v}$, $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

$f(x)$	$f'(x)$
$\sin kx$	$k\cos kx$
$\cos kx$	$-k\sin kx$
$\tan kx$	$k\sec^2 kx$
$\cot kx$	$-k\operatorname{cosec}^2 kx$
$\sec kx$	$k\sec kx \tan kx$
$\operatorname{cosec} kx$	$-k\operatorname{cosec} kx \cot kx$
e^{kx}	ke^{kx}
$\ln kx$	$\frac{k}{kx}$
a^{kx}	$k\ln a\, a^{kx}$
$f(y)$	$f'(y)\frac{dy}{dx}$
y^n	$ny^{n-1}\frac{dy}{dx}$
xy	$x\frac{dy}{dx} + y$

concave : $f''(x) \leq 0$

convex : $f''(x) \geq 0$

point of inflection = $f''(x)$ changes sign

Numerical Methods

If the function $f(x)$ is continuous on the interval $[a, b]$ and $f(a)$ and $f(b)$ have opposite signs then $f(x)$ has at least one root, x, which satisfies: $a < x < b$

To solve an equation of the form $f(x) = 0$ by an iterative method, rearrange $f(x) = 0$ into the form $x = g(x)$ and use the iterative formula $x_{n+1} = g(x_n)$.

Newton-Raphson formula: $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

Vectors

Distance from O to $(x, y, z) = \sqrt{x^2 + y^2 + z^2}$

Distance between (x_1, y_1, z_1) and $(x_2, y_2, z_2) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$

$\underline{i} = \begin{pmatrix}1\\0\\0\end{pmatrix}$, $\underline{j} = \begin{pmatrix}0\\1\\0\end{pmatrix}$, $\underline{k} = \begin{pmatrix}0\\0\\1\end{pmatrix}$

If the vector $a = x\underline{i} + y\underline{j} + z\underline{k}$ makes an angle θ_x with the positive x-axis then $\cos\theta_x = \frac{x}{|a|}$

Integration

$f(x)$	$\int f(x) \cdot dx$
x^n	$\frac{1}{n+1} x^{n+1} + c$
e^{kx}	$\frac{1}{k} e^{kx} + c$
$\frac{1}{x}$	$\ln \lvert x \rvert + c$
$\cos kx$	$\frac{1}{k} \sin kx + c$
$\sin kx$	$-\frac{1}{k} \cos x + c$
$\sec^2 kx$	$\frac{1}{k} \tan kx + c$
$\operatorname{cosec}^2 kx$	$-\frac{1}{k} \cot x + c$
$\sec kx \tan kx$	$\frac{1}{k} \sec kx + c$
$\operatorname{cosec} kx \cot kx$	$-\frac{1}{k} \operatorname{cosec} kx + c$
$f'(ax+b)$	$\frac{1}{a} f(ax+b) + c$
$\tan x$	$\ln \lvert \sec x \rvert + c$
$\cot x$	$\ln \lvert \sin x \rvert + c$
$\sec x$	$\ln \lvert \sec x + \tan x \rvert + c$
$\operatorname{cosec} x$	$-\ln \lvert \operatorname{cosec} x + \cot x \rvert + c$
$k \frac{f'(x)}{f(x)}$	TRY $\frac{d}{dx} \ln \lvert f(x) \rvert$
$k \ f'(x)(f(x))^n$	TRY $\frac{d}{dx}(f(x))^{n+1}$

By Parts: $\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} \cdot dx$

$\int f(x) - g(x) \cdot dx = \int f(x) dx - \int g(x) \cdot dx$

$\int_a^b y \cdot dx \approx \frac{1}{2} h (y_0 + y_n + 2(y_1 + y_2 \cdots))$

$h = \frac{b-a}{n}$

If struggling check part a)

↳ might have proved something you can use

By Parts: use when y = product of algebraic and trig.

product of algebraic and exponential

ln function

Substitution: use when y = product of two algebraic functions

↳ note: $\frac{du}{dx}$ should cancel out the other part

Reverse chain Rule: use when $y = f'(x)(f(x))^n$

ln Integration: use when $y = k \frac{f'(x)}{f(x)}$

Partial fractions: denominator is factorised into linear factors or is quadratic